# Self-consistency properties of elementary Reynolds stress closures

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# TOPICS

– Summary of paper on spherical harmonics expansion of the correlation tensor: Rubinstein, Kurien, Cambon; submitted to JoT.

– Reformulation of the Launder-Reece-Rodi model.

## SO(3): rotation group in 3D – action on polynomials

$$\underbrace{\{x, y, z\}}_{\text{3D}} = \underbrace{\{x, y, z\}}_{\text{3D: spin 1}}$$

$$\underbrace{\{x^2, xy, \cdots\}}_{\text{6D}} = \underbrace{\{x^2 + y^2 + z^2\}}_{\text{1D: spin 0}} + \underbrace{\{x^2 - y^2, y^2 - z^2, \cdots\}}_{\text{5D: spin 2}}$$

$$\underbrace{\{x^3, x^2y, \cdots\}}_{\text{10D}} = \underbrace{(x^2 + y^2 + z^2)\{(x, y, z)\}}_{\text{3D: spin 1}} + \underbrace{\{(x^3 - xy^2, x^3 - xz^2, \cdots\}}_{\text{7D: spin 3}}$$

$$3 = \tfrac{1}{2}2 \times 3 = 3 \quad 10 = \tfrac{1}{2}4 \times 5 = 3 + 7$$
$$6 = \tfrac{1}{2}3 \times 4 = 1 + 5 \quad 15 = \tfrac{1}{2}5 \times 6 = 1 + 5 + 9$$

Single point moments may oversimplify the description of anisotropy.

Spherical harmonics basis of 5D space $\{x^2 - y^2, y^2 - z^2, z^2 - x^2 \cdots\}$:

$$(x + iy)^2 = Y^{2,2} + iY^{2,-2} \quad (x + iy)z = Y^{2,1} + iY^{2,-1}$$
$$3z^2 - (x^2 + y^2 + z^2) = Y^{2,0} = Y^2$$

Spherical harmonics basis of 7D space $\{x^3 - 2xy^2, x^3 - 2xz^2, \cdots\}$:

$$(x + iy)^3 = Y^{3,3} + iY^{3,-3} \quad (x + iy)^2 z = Y^{3,2} + iY^{3,-2}$$
$$(x + iy)(5z^2 - 2(x^2 + y^2 + z^2)) = Y^{3,1} + iY^{3,-1}$$
$$5z^3 - 3z(x^2 + y^2 + z^2) = Y^{3,0} = Y^3$$

SO(3) alone does not provide any basis for these spaces.

Correlation tensor $U_{ij}(\boldsymbol{k})$ is solenoidal: $k_i U_{ij}(\boldsymbol{k}) = k_j U_{ij}(\boldsymbol{k}) = 0$.
For isotropy, $U_{ij}(\boldsymbol{k}) = U(k) P_{ij}(\boldsymbol{k})$ where $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k^{-2} k_i k_j$
Directional-polarization decomposition (Cambon):

$$U_{ij}(\boldsymbol{k}) = U_{ij}^{dir}(\boldsymbol{k}) + U_{ij}^{pol}(\boldsymbol{k})$$

where $U_{ij}^{dir}(\boldsymbol{k}) = \frac{1}{2}(\mathsf{U}(\boldsymbol{k}) : \mathsf{P}(\boldsymbol{k})) P_{ij}(\boldsymbol{k})$ and
$U_{ij}^{pol}(\boldsymbol{k}) = U_{ij}(\boldsymbol{k}) - U_{ij}^{dir}(\boldsymbol{k})$. Or

$$U_{ij}(\boldsymbol{k}) = \underbrace{U^{dir}(\boldsymbol{k}) P_{ij}(\boldsymbol{k})}_{\text{tensorially isotropic}} + \underbrace{U_{ij}^{pol}(\boldsymbol{k})}_{\text{trace-free solenoidal}}$$

For axial symmetry (Batchelor, Chandrasekhar)

$$U_{ij}(\boldsymbol{k}) = U^{dir}(\boldsymbol{k})P_{ij}(\boldsymbol{k}) + U^{pol}(\boldsymbol{k})S_{ij}^{pol}(\boldsymbol{k})$$

where $S_{ij}^{pol}(\boldsymbol{k}) =$

$$k^2 a_i a_j - (\boldsymbol{a}\cdot\boldsymbol{k})(k_i a_j + k_j a_i) + \tfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{k})^2\left[\delta_{ij} + k^{-2}k_i k_j\right] - \tfrac{1}{2}a^2 k^2 P_{ij}(\boldsymbol{k})$$

Spherical harmonics expansions:

$$U^{dir}(\boldsymbol{k}) = \sum_{\nu\geq 0,\ \text{even}} A_\nu(k)k^{-\nu}Y^\nu(\boldsymbol{k})$$

$$Y^0(\boldsymbol{k}) = 1, \quad Y^2(\boldsymbol{k}) = k^2 - 3(\boldsymbol{a}\cdot\boldsymbol{k})^2, \quad Y^4(\boldsymbol{k}) = 3k^4 - 30k^2(\boldsymbol{a}\cdot\boldsymbol{k})^2 + 35(\boldsymbol{a}\cdot\boldsymbol{k})^4$$

with

$$U^{pol}(\boldsymbol{k}) = \sum_{\nu\geq 2\ \text{even}} B_\nu(k)k^{-\nu}Z^\nu(\boldsymbol{k})$$

$$Z^2(\boldsymbol{k}) = 7(\boldsymbol{a}\cdot\boldsymbol{k})^2 - k^2 \quad Z^4(\boldsymbol{k}) = 33(\boldsymbol{a}\cdot\boldsymbol{k})^4 - 18k^2(\boldsymbol{a}\cdot\boldsymbol{k})^2 + k^4$$

## general anisotropy: even spin polarization

$$U_{ij}^{pol}(\boldsymbol{k}) = \sum_{\nu \geq 0 \text{ even}} \sum_{-\nu \leq \mu \leq \nu} A_{\nu,\mu}(k) Y_{ij}^{\nu,\mu}(\boldsymbol{k})$$

Angular dependence parametrized by $Y_{ij}^{\nu,\mu}(\boldsymbol{k})$; amplitudes $A_{\nu,\mu}(k)$ depend only on *wavenumber*.

$$Y_{ij}^{\nu,\mu}(\boldsymbol{k}) = \mathcal{L}_{ij}^{\nu}[Y^{\nu,\mu}(\boldsymbol{k})]$$

for rotation invariant (spin 0) matrices of differential operators $\mathcal{L}_{ij}^{\nu}$ (Arad et al., Zemach, ....)

$$
\begin{aligned}
&Y_{ij}^{\nu,\mu}(\boldsymbol{k}) + iY_{ij}^{\nu,-\mu}(\boldsymbol{k}) \\
&= \mu(\mu-1)(k_x+ik_y)^{\mu-2}\left(\frac{\partial^{\mu}}{\partial k_z^{\mu}}Y^{\nu}(\boldsymbol{k})\right)\left(Y_{ij}^{2,2}(\boldsymbol{k}) + iY_{ij}^{2,-2}(\boldsymbol{k})\right) \\
&+2\mu(k_x+ik_y)^{\mu-1}\left(\frac{\partial^{\mu+1}}{\partial k_z^{\mu+1}}Y^{\nu}(\boldsymbol{k})\right)\left(Y_{ij}^{2,1}(\boldsymbol{k}) + iY_{ij}^{2,-1}(\boldsymbol{k})\right) \\
&+(k_x+ik_y)^{\mu}\left(\frac{\partial^{\mu+2}}{\partial k_z^{\mu+2}}Y^{\nu}(\boldsymbol{k})\right)Y_{ij}^{2,0}(\boldsymbol{k})
\end{aligned}
$$

## odd spin polarization

– Rotational strains couple even and odd spins.
– Similar formulas, but different basis tensors $\mathsf{X}^{2,\mu}(\boldsymbol{k})$.
– Kassinos et al. structure tensor formalism: *stropholysis* is related to spin 3 polarization. The ‘stropholysis spectrum’

$$Q_{ij\ell}(k) = \epsilon_{ipq} \oint dS(\boldsymbol{k})\ U^{pol}_{jq}(\boldsymbol{k}) k^{-2} k_p k_\ell$$

defines the projection of U onto its spin 3 component.

## Reformulation of the LRR model

Mean flow couplings:

$$\dot{U}_{ij}(\boldsymbol{k}) = -\underbrace{\left[U_{ip}(\boldsymbol{k})\frac{\partial U_j}{\partial x_p} + U_{jp}(\boldsymbol{k})\frac{\partial U_i}{\partial x_p}\right]}_{\text{production}} + \underbrace{k_m \frac{\partial}{\partial k_n} U_{ij}(\boldsymbol{k})\frac{\partial U_m}{\partial x_n}}_{\text{mean}-\text{flow distortion}}$$

$$+ \underbrace{2k^{-2}\left[k_i k_m U_{pj}(\boldsymbol{k}) + k_j k_m U_{pi}(\boldsymbol{k})\right]\frac{\partial U_m}{\partial x_p}}_{\text{rapid pressure}-\text{strain}}$$

Single-point reduction:

$$\int d\boldsymbol{k}\ \dot{U}_{ij}(\boldsymbol{k}) = -\underbrace{\int d\boldsymbol{k}\ U_{ip}(\boldsymbol{k})}_{\text{OK, `closed`}}\frac{\partial U_j}{\partial x_p} + \int d\boldsymbol{k}\ \underbrace{2k^{-2} k_i k_m U_{pj}(\boldsymbol{k})}_{\text{closure problem}}\frac{\partial U_m}{\partial x_p}$$

Directional-polarization decomposition

$$\dot{U}^{dir}(\boldsymbol{k}) = -U^{dir}(\boldsymbol{k})(\mathsf{P}(\boldsymbol{k}) : \mathsf{S}) - \mathsf{U}^{pol}(\boldsymbol{k}) : \mathsf{S} + k_m \frac{\partial}{\partial k_n} U^{dir}(\boldsymbol{k}) \frac{\partial U_m}{\partial x_n}$$

$$\begin{aligned}
&\dot{U}^{pol}_{ij}(\boldsymbol{k}) \\
&= -\tfrac{1}{2} U^{dir}(\boldsymbol{k}) \Big( P_{im}(\boldsymbol{k}) P_{jn}(\boldsymbol{k}) + P_{in}(\boldsymbol{k}) P_{jm}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k}) P_{mn}(\boldsymbol{k}) \Big) \frac{\partial U_m}{\partial x_p} \\
&- \Big( U^{pol}_{ip}(\boldsymbol{k}) P_{jm}(\boldsymbol{k}) + U^{pol}_{jp}(\boldsymbol{k}) P_{im}(\boldsymbol{k}) - U^{pol}_{mp}(\boldsymbol{k}) P_{ij}(\boldsymbol{k}) \Big) \frac{\partial U_m}{\partial x_p} \\
&+ k_m \frac{\partial}{\partial k_n} U^{pol}_{ij}(\boldsymbol{k}) \frac{\partial U_m}{\partial x_n} + k^{-2} \Big( k_i k_m U^{pol}_{pj}(\boldsymbol{k}) + k_j k_m U^{pol}_{pi}(\boldsymbol{k}) \Big) \frac{\partial U_m}{\partial x_p}
\end{aligned}$$

Two models for the correlation tensor consistent with LRR:

$$U_{ij}(\boldsymbol{k}) = \tfrac{1}{2}U(k)P_{ij}(\boldsymbol{k}) + R_{ij}(\boldsymbol{k})$$

where, for model I,

$$R_{ij}(\boldsymbol{k}) = k^{-2}(\mathsf{H}(k) : \boldsymbol{kk})P_{ij}(\boldsymbol{k})$$

and for model II,

$$\begin{aligned} &R_{ij}(\boldsymbol{k}) = H_{ij}(k) - k^{-2}k_m(k_i H_{mj}(k) + k_j H_{mi}(k)) \\ &+\tfrac{1}{2}k^{-2}(\delta_{ij} + k^{-2}k_i k_j)(\mathsf{H}(k) : \boldsymbol{kk}) \end{aligned}$$

The Reynolds stress deviator is

$$R_{ij} = \int d\boldsymbol{k}\ R_{ij}(\boldsymbol{k}).$$

## model I

Dir-pol

$$U^{dir}(\boldsymbol{k}) = U(k) + k^{-2}\mathsf{H}(k) : \boldsymbol{kk}$$
$$U^{pol}_{ij}(\boldsymbol{k}) = 0$$

Stress deviator

$$R_{ij} = \frac{2}{15}\int_0^\infty k^2 dk \; H_{ij}(k)$$

Tensor H is basically Kassinos et al. *dimensionality*.

## LRR and model I

Mean-flow coupling, directional anisotropy:

$$\dot{U}(k) + k^{-2}\dot{\mathsf{H}}(k) : \boldsymbol{kk} = U(k)k^{-2}(\mathsf{S} : \boldsymbol{kk}) + k^{-4}(\mathsf{H}(k) : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk})$$
$$+k_m \frac{\partial}{\partial k_n} U(k) \frac{\partial U_m}{\partial x_n} + k_m \frac{\partial}{\partial k_n} k^{-2}(\mathsf{H}(k) : \boldsymbol{kk}) \frac{\partial U_m}{\partial x_n}$$

Evaluating the derivatives:

$$\dot{U}(k) + k^{-2}\dot{\mathsf{H}}(k) : \boldsymbol{kk} = \Big(U(k) + kU'(k)\Big)k^{-2}\mathsf{S} : \boldsymbol{kk}$$
$$-k^{-4}(\mathsf{H}(k) : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk}) + k^{-3}(\mathsf{H}'(k) : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk})$$
$$+k^{-2}\Big(\mathsf{H}(k) \cdot \mathsf{S} + \mathsf{S} \cdot \mathsf{H}(k)\Big) : \boldsymbol{kk} + k^{-2}\Big(\mathsf{H}(k) \cdot \Omega - \Omega \cdot \mathsf{H}(k)\Big) : \boldsymbol{kk}$$

Spin decomposition:

$$\underbrace{\dot{U}(k)}_{\text{spin } 0} + \underbrace{k^{-2}\dot{\mathsf{H}}(k) : \boldsymbol{kk}}_{\text{spin } 2} = \underbrace{\Big(U(k) + kU'(k)\Big)k^{-2}\mathsf{S} : \boldsymbol{kk}}_{\text{spin } 2}$$

$$- \underbrace{k^{-4}(\mathsf{H}(k) : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk})}_{\text{spins } 2,3,4} + \underbrace{k^{-3}(\mathsf{H}'(k) : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk})}_{\text{spins } 2,3,4}$$

$$+ \underbrace{k^{-2}\Big(\mathsf{H}(k) \cdot \mathsf{S} + \mathsf{S} \cdot \mathsf{H}(k)\Big) : \boldsymbol{kk}}_{\text{spins } 0,2} + \underbrace{k^{-2}\Big(\mathsf{H}(k) \cdot \Omega - \Omega \cdot \mathsf{H}(k)\Big) : \boldsymbol{kk}}_{\text{spin } 2}$$

$$(\mathsf{H} : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk}) = \|\mathsf{A}\|_4 + \|\mathsf{A}\|_2 + \|\mathsf{A}\|_0$$

where

$$\|\mathsf{A}\|_4 = (\mathsf{H} : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk}) - \tfrac{2}{7}k^2(\mathsf{H} \cdot \mathsf{S} + \mathsf{S} \cdot \mathsf{H}) : \boldsymbol{kk} + \tfrac{2}{35}k^4\mathsf{H} : \mathsf{S}$$
$$\|\mathsf{A}\|_2 = \tfrac{2}{7}k^2\Big(\mathsf{H} \cdot \mathsf{S} + \mathsf{S} \cdot \mathsf{H} - \tfrac{2}{3}(\mathsf{H} : \mathsf{S})\mathsf{I}\Big) : \boldsymbol{kk}$$
$$\|\mathsf{A}\|_0 = \tfrac{2}{15}k^4(\mathsf{H} : \mathsf{S})$$

Equate terms of equal spin. For spins 0 and 2,

$$\dot{U}(k) = \tfrac{8}{15}(\mathsf{H} : \mathsf{S}) + \tfrac{2}{15}k(\mathsf{H}' : \mathsf{S})$$

$$\dot{\mathsf{H}}(k) = \Big(U(k) + kU'(k)\Big)\mathsf{S} + \tfrac{5}{7}\Big(\mathsf{H}(k)\cdot\mathsf{S} + \mathsf{S}\cdot\mathsf{H}(k) - \tfrac{2}{3}(\mathsf{H}(k) : \mathsf{S})\mathsf{I}\Big)$$

$$+\tfrac{2}{7}k\Big(\mathsf{H}'(k)\cdot\mathsf{S} + \mathsf{S}\cdot\mathsf{H}'(k) - \tfrac{2}{3}(\mathsf{H}'(k) : \mathsf{S})\mathsf{I}\Big) - \Big(\mathsf{H}(k)\cdot\Omega - \Omega\cdot\mathsf{H}(k)\Big)$$

Energy equation:

$$\dot{\mathsf{k}} = \int_0^\infty dk\ k^2\dot{U}(k) = \int_0^\infty dk\ \Big(\tfrac{8}{15}k^2(\mathsf{H} : \mathsf{S}) + \tfrac{2}{15}k^3(\mathsf{H}' : \mathsf{S})\Big) = -\mathsf{R} : \mathsf{S}$$

Stress deviator equation:

$$\dot{\mathsf{R}} = \tfrac{4}{15}\mathsf{k}\mathsf{S} - \tfrac{1}{7}\Big(\mathsf{R}\cdot\mathsf{S} + \mathsf{S}\cdot\mathsf{R} - \tfrac{2}{3}(\mathsf{R} : \mathsf{S})\mathsf{I}\Big) + \Big(\mathsf{R}\cdot\Omega - \Omega\cdot\mathsf{R}\Big)$$

This is an LRR model with a special choice of constants.

However, the spectral equation is not satisfied: spin 4 in the dir equation, and spin 2 in the pol equation are both left over.

Model I cannot satisfy the mean flow coupling equations exactly.

It is a good *approximation* provided

$$(\mathsf{H} : \boldsymbol{kk})(\mathsf{S} : \boldsymbol{kk}) - \tfrac{2}{7}k^2(\mathsf{H}\cdot\mathsf{S} + \mathsf{S}\cdot\mathsf{H}) : \boldsymbol{kk} + \tfrac{2}{35}k^4\mathsf{H} : \mathsf{S} \approx 0$$

and

$$\Big(U(k) + k^{-2}(\mathsf{H} : \boldsymbol{kk})\Big) \times$$
$$\Big(P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k}) + P_{in}(\boldsymbol{k})P_{jm}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k})P_{mn}(\boldsymbol{k})\Big)S_{mn} \approx 0$$

Application of results to assess accuracy/breakdown of LRR model: ‘in progress’ (at LANL).

– Results for model II are similar, but spin decompositions are not so simple: decomposition of tensor $(\mathsf{S} : \boldsymbol{kk})\mathsf{H}$ was done by Zemach.
– 'Better' models: including descriptors of higher spins gives a hierarchy of differential equations.
– Although stress evolution is determined by spins of all orders, stress itself is determined by spin 2 alone.
– It suggests replacing evolution equations for higher spins by algebraic relations leading to a *normal solution*: an approximate solution of the stress evolution equations in terms of stress-determining quantities only.